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LETTER TO THE EDITOR

Position-space renormalisation group for directed branched polymers

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Abstract. All previous attempts to obtain accurate quantitative estimates of critical properties for directed percolation or directed animals with position-space renormalisation group (PSRG) have failed. We analyse the problems that appear in the renormalisation of directed models, and then present a PSRG that avoids these problems and gives reliable predictions.

The PSRG has in the past been applied to directed geometrical problems such as directed percolation and directed lattice animals (see e.g. the recent review by Stanley *et al* (1982)). Although reasonable qualitative phase diagrams were obtained, difficulties appeared which could not be handled. In this letter we propose a method that avoids these difficulties. We illustrate the method for obtaining accurate quantitative estimates using directed site animals as an example.

It is well known that directed problems have two different length scales, one in the 'time' direction ($\|$) and one ($\perp$) for the $(d-1)$ remaining directions

$$\xi_\| \sim N^{\nu_\|}, \qquad \xi_\perp \sim N^{\nu_\perp} \tag{1}$$

where $\nu_\| \neq \nu_\perp$ and N is the number of occupied sites (or bonds). The usual PSRG approaches do not take into account the different length scales and therefore yield incorrect exponents. This remark applies to the approach of Redner and Yang (1982) for the case of the two-dimensional animals and to that of Oliveira (1983) for two-dimensional percolation (in this regard, see also the reply by Phani and Dhar (1982)). The same problem occurs in PSRG of mixtures of resistors and diodes, at least in the limit where there are only diodes that conduct in one direction (Redner and Brown 1981, Redner 1981, 1982a, b, Dorogovtsev 1982).

To visualise the problem, figure 2 shows different results obtained by the usual PSRG. For directed animals we plot the exponent of the parallel correlation length $\nu_\|$ against $(\ln b)^{-1}$, where b is the rescaling length: the bond animal result of Redner and Yang (1982) with the rule of figure 1(a), site animals with the rule of figure 1(b) and site animals with the corner-to-corner rule of figure 1(c) and a renormalisation factor $b\sqrt{2}$. Clearly none of the renormalisations yields results that tend towards

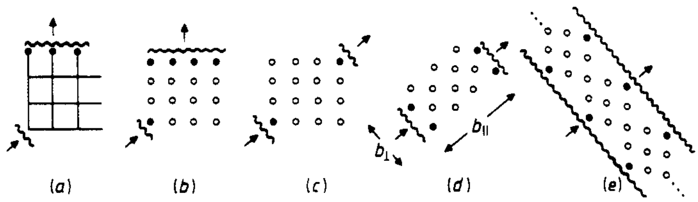

Figure 1. Rules for the renormalisation. A renormalised cell is occupied if a spanning cluster extends from: (a,b) the lower left corner to either the upper border or the right border; (c) the lower left corner to the upper right corner; (d) the lower left border to the upper right border; (e) any of the lower left corners to any of the upper right corners.

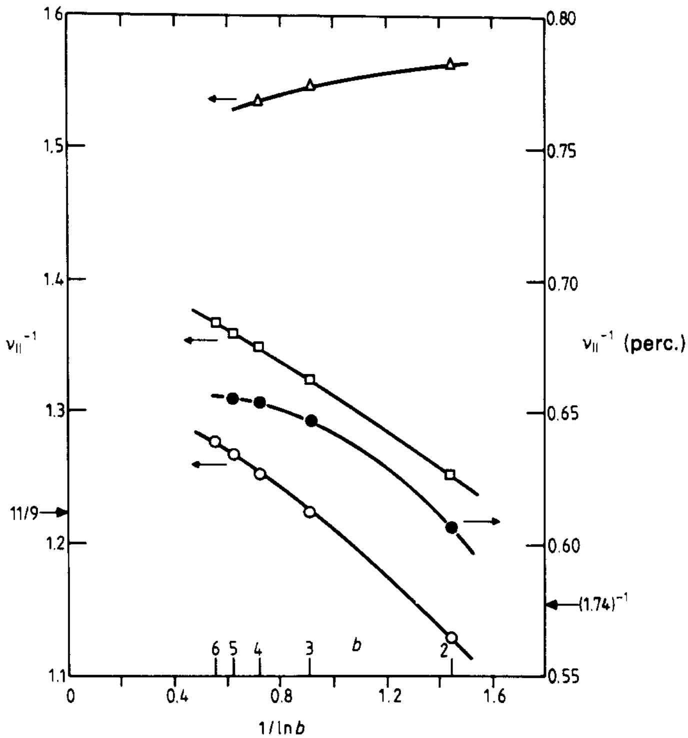

Figure 2. Exponent $\nu_\parallel$ against $1/\ln b$. Left axis for directed animals: △ bond animals from Redner and Yang (1982); □ site animals with the rule of figure 1(b); ○ site animals with the rule of figure 1(c). Right axis for directed percolation: ● site percolation with the rule of figure 1(b).

the value $\nu_\parallel = \frac{9}{11}$, which was recently calculated using phenomenological renormalisation (Nadal *et al* 1982)†. In figure 2 we also show directed-site percolation data using the rule of figure 1(b). Again the numbers clearly do not tend towards the expected value which is about $\nu_\parallel = 1.74$ (Kinzel and Yeomans 1981).

† In this context one should point out that also Redner and Yang (1982) calculated $\nu_\parallel$ by extrapolation of exact series calculations and found it to be 0.800 ± 0.001 and clearly not $\frac{9}{11}$. This discrepancy is not yet completely understood. Also, the value $\nu_\parallel = 0.80$ is far off the PSRG results of figure 2.

One source of the problems occurring in PSRG of directed systems is the following (Phani and Dhar 1982, Cardy 1982). Since there are two correlation lengths which scale differently with N, one must use two different rescaling factors $b_\parallel$ and $b_\perp$ in the PSRG in such a way that

$$b_\parallel \simeq b_\perp^{\nu_\parallel/\nu_\perp}. \tag{2}$$

Since $\nu_\parallel \neq \nu_\perp$ this means that the shape of the cell must change in a definite way at each renormalisation step. This can be done in the case of site animals by cells of the type shown in figure 1(d). Since $\nu_\parallel/\nu_\perp$ is not known, it must be obtained self-consistently. Note that only certain ratios of $b_\parallel/b_\perp$ can be realised with the small cell sizes that one can calculate exactly. Thus this method, although in principle correct, turns out to be not very feasible.

The problem of asymmetric rescaling and the change of the cell shape can possibly be avoided if we make one of the two lengths $b_\parallel$ or $b_\perp$ infinite. This is realised in the phenomenological renormalisation calculations of Nadal *et al* (1982) and Kinzel and Yeomans (1981). The question is how to realise an infinite length if only *finite* cells (as opposed to *infinite* strips) are available.

We propose the renormalisation scheme sketcted in figure 1(e). Explicitly, we calculate the weight of an $n \times n$ cell:

$$R_n(x) = \sum_{2^N \text{conf}} x^N, \tag{3}$$

where x is the fugacity to occupy one site and the sum goes over all configurations that connect one corner with the opposite as in the rule of figure 1(c). N is the number of occupied sites in a given configuration. We then patch the squares together to an infinitely long sequence as shown in figure 1(e). Thus we additionally allow, from each corner, clusters that connect to corners of neighbouring cells. In the directed case there are only three such clusters. If one renormalises from $n \times n$ to $m \times m$ cells, the fixed point $x_0(n,m)$ of the equation

$$[1/(n-1)][R_n(x') + 2x'^{2n-1} + x'^{4n-3}] = [1/(m-1)][R_m(x) + 2x^{2m-1} + x^{4m-3}] \tag{4}$$

is an approximation for the critical fugacity which in the limit $n,m \to \infty$ becomes exact. The critical exponent $\nu_\parallel$ is approximated by

$$[\nu_\parallel(n,m)]^{-1} = \ln(\lambda_n/\lambda_m)/\ln(n-1/m-1) - 1, \tag{5}$$

with

$$\lambda_n = R'_n(x_c) + 2(2n-1)x_c^{2(n-1)} + (4n-3)x_c^{4(n-1)}, \tag{6}$$

evaluated at the fixed point of (4).

We calculate $R_n(x)$ by counting on a square lattice the configurations, using an algorithm due to Martin (1974) up to $n = 6$; we obtain x_c and $\nu_\parallel$ from (4)–(6). In figure 3 we plot $x_c(n, n-1)$ against $1/n^3$ and $1/(n-1)^3$ and see that the presumably exact value $x_c = \frac{1}{3}$ (Dhar *et al* 1982, Dhar 1982) is convincingly approached. We choose the inverse cube of the system size as axis because it gives the straightest line; we have no theoretical background to justify this choice. In figure 4 we plot $\nu_\parallel(n, n-1)$ against $1/n^2$ and $1/(n-1)^2$. Again here the value $\nu_\parallel = \frac{9}{11}$ obtained by Nadal *et al* (1982) is approached.

We conclude that, using the trick of patching square cells together to an infinite sequence, one can avoid the problem of changing cell shapes and obtain a correct

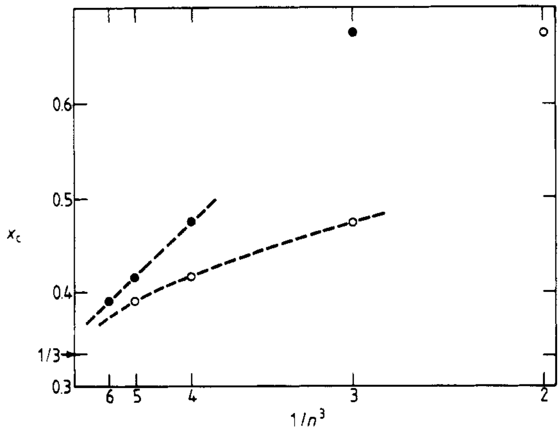

Figure 3. Critical fugacity $x_c(n, n-1)$ against $(1/n^3)$ (●) and $[1/(n-1)^3]$ (○).

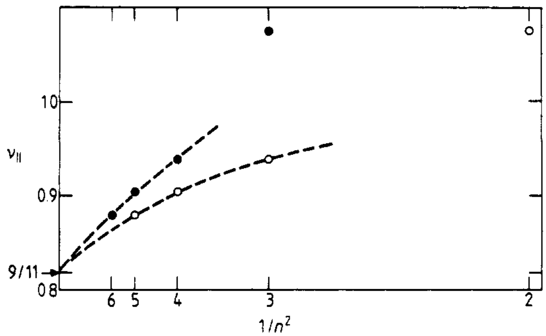

Figure 4. Correlation length exponent parallel to the 'time' direction $(n, n-1)$ against $(1/n^2)$ (●) and $(1/(n-1)^2]$ (○).

renormalisation prescription. Such strings of cells should also be used for renormalisations with several coupling constants such as those used to calculate the conductivity of random mixtures of diodes and resistors. We also suggest the application of our approach for other problems with direction-dependent correlation lengths, as is the case for Lifshitz points (Hornreich *et al* 1975).

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